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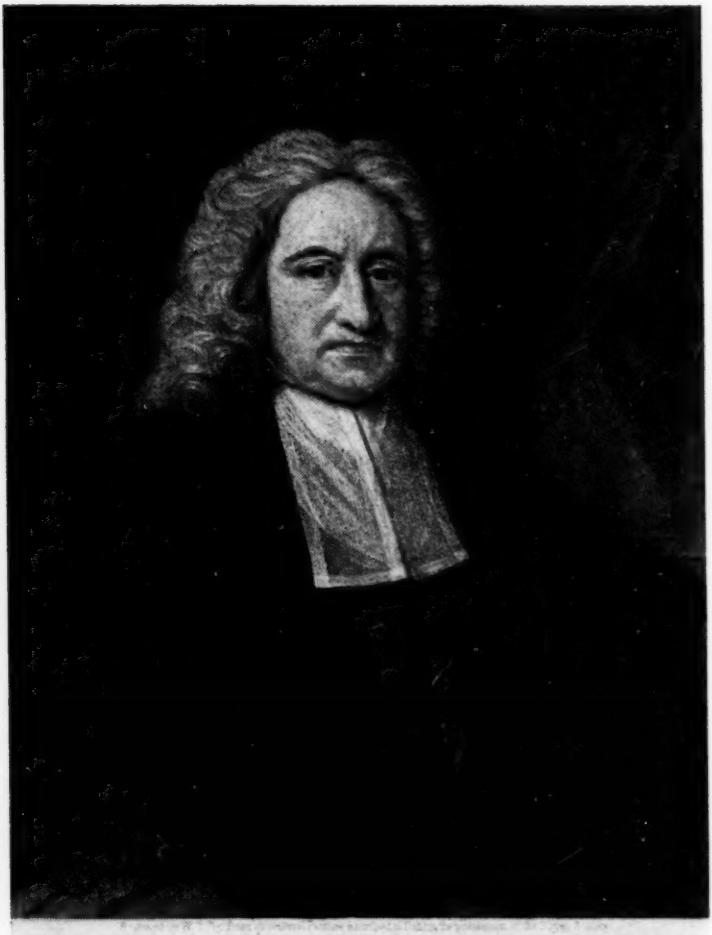

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THE MATHEMATICS TEACHER

Volume XXVI Number 4

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Adapting Instructional Material to Individual Differences in Learning

By J. T. JOHNSON
Chicago Normal College, Chicago, Illinois

SO FAR THERE have been two main teaching methods of dealing with the problem of individual differences in learning, the homogeneous grouping of classes on the basis of ability and the individual instruction method. The earlier attempts at homogeneous grouping were known as the Batavia plan and the Pueblo plan in which groups of students within a class were taught and promoted as groups. It was not until the advent of the intelligence tests that segregation into class groups on the basis of ability began. We now have the well known A, B and C groups or the X, Y and Z groups in many high schools.

The individual instruction method from printed material is more recent in its development and is now known as the Dalton plan, the contract plan, the Winnetka plan and the McDade plan. All of these are characterized by individual progress and promotion.

THE DIFFERENTIATED SUBJECT-MATTER METHOD

It is my purpose here to describe a third plan which has been tried and found successful at the Chicago Normal College during the past

six years. For want of a better name I have called it the differentiated subject-matter plan.

I want to give first some reasons for its use then describe the procedure by stages and later give some advantages it offers.

I. Arguments For Its Use

I am going to read here some sentences that bear upon this topic. Some are quotations and others are extracts gleaned from notes from lectures at Teachers College last year.

"In a democracy the need for developing leaders is very essential."

"We need new and different material for brighter students."

"The incoming poorer classes now demand a different treatment."

"Let the slow moving mind have a chance. The quick alert mind is given recognition in the earlier mastery."

"Students in mathematics can not begin at any place and go on as they can in history, English or social studies because of the unique nature of mathematics."

"Help the student when he gets into difficulties."—Judd

"Business men say that boys and girls are not accurate enough in arithmetic."

These statements all argue for differentiated subject-matter. In this day of increasing hordes entering our high schools we are faced by a dilemma. If we are to keep up our old standards of high grade work and efficiency we must fail a greater number of students than formerly, and if we do not want to increase the number of failures we must lower our standards. I am afraid we have indulged too much in a lowering of standards in the recent past, writing soft courses in mathematics with the excuse that these students come to our classes and it is up to us to give them some kind of mathematics that they can learn and from which they can profit. What about the student that loves mathematics for its own sake, and who likes it the more, the more abstract it is? Should he not be taken care of although his kind may be only small in number?

Let us look at it from another angle. Let us ask any teacher of mathematics in this country today this question, Upon which half of the class do you spend most of your time and energy, the upper or lower in intelligence? The answer would be in 90% of the cases the lower half. If I ask this audience this question what would be your answer? This is followed by another question, Which half will bring the

greater return to society from their teaching? The answer is only too plain.

Under the existing conditions with the same subject-matter for all classes, the teacher with a load of forty to fifty students spends her energy on the dull students letting the bright ones shift for themselves in order to get a fair portion of her students through the course. She can do nothing else. It would be better if she spent her time on the bright student letting the dumb bells shift for themselves. This, however, she dare not do. Surely something ought to be done about this.

It is the thesis of this discussion that instructional material can be so analysed and arranged and the teaching procedure so geared that the dull and slow students will master certain essentials of the work at the same time that the brighter students will go into some more difficult and interesting work from which they will profit.

Many of the so-called dull students are ofttimes not really dull but only slow. Hence a part of the problem is to give them more time than the faster students.

II. How Material Is Differentiated

The subject matter is differentiated into three levels of difficulty known as set C, set B, and set A.

The C set is the minimum essential set and should contain just as much of the topic under consideration as we can justify will be of benefit to the student in later life and also give a foundation for future mathematics. It may be informational, social or mechanical. The many surveys such as those made by Charters, Woody, Wilson and Wise could perhaps be used as valuable criteria here. Great care needs be taken as to what goes into this set, more so than in the others because we demand an absolute mastery of this set by every individual before credit is given. Our criterion for putting in this set what we do put in should be logically sound at least.

Set B should contain more abstract material. It is taken voluntarily by those students who like mathematics. All that body of material which will be useful in future mathematics or science should be put here, even unto complex fractions. The students who like mathematics for its own sake get into this set. The majority of our future teachers of mathematics belong here.

Set A is for the near genius and the genius class. We have a limited few of them in most of our classes. The character of the exercises and

problems in this set should be such as to require originality in their solutions. "To draw a circle through two points and tangent to a given line" belongs here in its appropriate unit. Puzzle problems might go into this set if the topic is of such a nature as not to lend itself to application. The work should be difficult enough to challenge the brightest minds in the class. New applications for mathematical principles should be worked out here.

What should go into each unit and the method of differentiating the material within each unit can easily constitute a new field of research.

Perhaps the three levels, the Mechanistic, the Thinking and the Higher Mental levels which have been discussed could in some way be exemplified in these sets.

Set C should not attempt to prepare for college. Set B might well do so. Set A would then be left unhampered by neither college entrance requirements nor mechanical details but would be reserved as a place where the imagination of mathematics could feast at will and develop embryo Michelsons and Einsteins.

Each unit is organized as follows:

Introduction
Set C
  Presentation
  Practice exercise I
  Practice exercise II
  Test I
  Test II
  Test III
Set B
Set A

This organization is made clearer from a study of the next section.

III. Teaching-Procedure by Stages

(More fully elucidated on board by diagram)

		Time for teaching
First stage—	Introduction	(1 day)
Second stage—	Presentation	(2 to 5 days)
Third stage—	1st day—Practice Exercise I (Taken by all students)	(1 day)

—2nd day—Practice Exercise II (1 day)
(By those that had difficulty in Exercise I)

Fourth stage— Test I. (1 day)
(Taken by all)

Fifth stage— 1st day—Corrective work (1 day)
(*For* those who did not complete Test I, 100% correct and *by* those who did)
followed—by Test II
(Taken by those who did not get Test I 100% correct)
and set B
(By those who did Test I, 100% correct)
—2nd day—More intensive corrective work (1 day)
(*For* those who did not do Test II, 100% correct and *by* those who did)
followed by Test III
(By those who did not do Test II correctly)
and set B
(By those who did Test II correctly)
and set A
(By those who did Test I, 100% correct)

Sixth stage— Conference period— (1 day)
(In which questions on points not understood by some are asked and answered by other members of the class) Total time—(9 to 12 days)

IV. Method of Grading Results

(Also explained by diagram)

It was the intention, in labelling the sets, C, B and A, that students who completed set C which is the minimum requirement, would earn a credit of "C". Those who completed sets B and A would earn credits of "B" and "A" respectively. Parts of sets B and A could, of course, be rated C, B—, B and A—. Where grades are required in % as is done in Chicago Normal College, we give 80 for completion of C, 90 for completion of C and B, and 100 for completion of the three sets which means everything in the unit. No daily checking is required as but one check is placed after each name as the Test I, II, or III

is correctly completed. This and the B and A record are given during the progress of the unit which is only once in two or three weeks.

V. Comparison With the Individual Instruction Method

The essential difference between this method and the strictly individual method is that whereas in the individual method the brighter students go ahead at a faster pace into new work; in this method the brighter students go deeper and deeper into the same topic. In the individual method the differentiation is on the basis of quantity while here it is on the basis of quality. In the individual method the students are not together after the first week; while here they start together at the beginning of every unit and stay together except on two days in each unit during the fifth stage.

This method aims at keeping the essential benefits of the individual system and at the same time not destroy the class organization. It is felt that there are certain advantages accruing from the group work that should be retained besides saving time to the teacher as a new topic is to be presented.

VI. Comparison With the Homogeneous Grouping Plan

The main difference between the homogeneous grouping plan and this is that in the former the students are placed in C, B or A groups by the administration; whereas in this plan the students place themselves in C, B or A groups according to the merit of their work. It is felt that it is a bit more democratic and fairer to the student to put him on an equal footing and opportunity with the rest at the beginning of each unit rather than label him with a mark of C, B or A beforehand. The danger of producing snobs on the one hand and inferiority complexes on the other is thus avoided. No hard feelings are entertained by the student when he is given an equal chance with the rest although he cannot make the grade that some of the rest do. It is very much the same as in the athletic contest or track meet. The one that cannot clear the bar at 13′-6″ but sees his classmate do it does not complain as long as he has had his two trials at it. It is only when he is not allowed the same opportunity with the rest that the stigma of inferiority is felt.

It should be said that the homogeneous grouping method is very helpful if used rightly and furthermore it does not prevent the use of the plan under discussion because there is enough difference be-

tween the members of any group even if differentiated on basis of initial ability. In fact it is being used in such groups now.

Conclusion

Some Observed Advantages in this Method

1. The feeling of success is the best promoter of interest. The student likes to do that in which he succeeds. As enough time is given the slow student he grows in self-confidence.
2. As units are so prepared that what is a prerequisite in one unit is mastered in a foregoing unit, students begin each unit with the proper preparation.
3. The problem of discipline is reduced to a minimum, for each student is busy with work on his own level at every moment. The brighter students are challenged with work that requires their best efforts.
4. It reduces careless errors because each student is held to 100% mastery on every test before he can go on earning extra credit.
5. It induces the best attention during presentation. This is due to the eagerness of all not to miss anything in order to get 100% on the first test.
6. It reduces failures in that the slower students are helped in corrective work and given more than one test. It provides as many as four different teachings on any one difficult point in any one unit.
7. It has the advantage of the individual instruction methods without breaking up the organization of the class. It thus reduces the load and worry on the part of the teacher.
8. It is thoroughly democratic in that it gives every one the same chance in every unit.

Greetings to the Teachers of America

My message to you is, be Courageous! I have lived a long time. I have seen history repeat itself again and again. I have seen many depressions in business. Always, America has emerged from these stronger and more prosperous. Be as brave as your fathers before you were. Have faith. Go forward!—Thomas Edison.

The Problem of Mathematical Statistics

By A. R. CRATHORNE
University of Illinois, Urbana, Illinois

A STATISTICAL INQUIRY is usually made up of three parts, first the collection of the data, second, the analysis of the data and third, its interpretation for the particular purpose in view. The first and third stages depend almost entirely upon the field of investigation. Biological data should be collected and interpreted by a biologist. The collection of economic data should be planned by economists and interpreted by one well acquainted with the field of inquiry. The second stage on the other hand, is in general common to all fields. Whatever may be the subject matter the mathematical analysis has in a great degree the same essential features. Lack of appreciation of this three-fold problem of statistics has led to some extremely doubtful results in statistical analysis. No amount of mathematical training and ability can take the place of the judgment and common sense that comes from a thorough knowledge of the field in which the problem lies. On the other hand much time and energy may be wasted in the analysis of data by lack of understanding of the fundamentals of the mathematical processes used in the investigation. This is usually exhibited in the blind use of formulas without realization of the assumptions and limitations used in their derivation.

The field of this second stage in a statistical problem is usually called mathematical statistics or theoretical statistics. When one wishes to emphasize the fact that a branch of knowledge is well organized or wants to summarize it in a few words, it has become rather customary to speak of its problem. Thus we have the problem of mathematical statistics. Many men have stated it. In its most concise form the problem of this part of statistics is stated to be the reduction of large masses of data to a few quantities each of which is often called a statistical constant, or more recently a statistic. For example the daily temperature records of a weather station may be summarized in two statistics, one being the mean temperature and the other, one of the many measures of variation from the mean. These two constants give a great deal of information about the weather in the neighborhood of the station. The

English statistician, R. A. Fisher states the problem somewhat more definitely thus: "A quantity of data which usually by its mere bulk is incapable of entering the mind, is to be replaced by relatively few quantities which shall adequately represent the whole, or which in other words shall contain as much as possible, ideally the whole, of the relevant information contained in the original data." Karl Pearson says that the fundamental problem of statistics is as follows: "An event has occurred p times out of n trials where we have no a priori knowledge of the frequency of the event in the total population of occurrences. What is the probability of its occurring r times in s trials." Von Mises, the leading German statistician, says in substance that our problem consists in devising the contents of an urn, drawings from which give results which conform to the data at hand. A little examination of these three statements will show that they are not far apart.*

It is not my intention to formulate another statement of the problem but to give a sort of outline of the processes of reduction of the data to a few statistics. We start with any large mass of data. To make our language concrete, let each observation, or each set of related observations be written on a card. We can make two kinds of piles of cards. In the first, the cards are ordered singly or in groups according to some attribute, usually time or space. For example, our data may be the weekly index of prices for a series of years arranged in order of time. This ordered arrangement of data leads to the problems of determination of trend, measurement and elimination of seasonal variation, the discovery of cycles and lags, and the questions of regression and correlation peculiar to this class of data. Closely in touch with problems of this nature is the so-called Lexian theory. We all know that in any country the death rate will vary from district to district. Even if the underlying probability of death were the same for all parts of the country, the tabulated statistics would vary from county to county, the fluctuations being due to chance. The Lexis criterium gives a means of telling whether these differences as actually reported can be explained away as chance fluctuations, or whether the cause is to be sought elsewhere. Variations of infant mortality from state to state in the United States are such as to preclude any assumption of a

* This paragraph and the two following are taken from the author's vice-presidential address before the American Statistical Association, Dec. 29, 1930.

constant underlying probability of life or death for infants. Another theory arising out of data thus ordered is the seeming paradox of the antagonistic relations between homogeneity and stability as presented in recent years by Bortkiewicz, the German economist. Ordinarily we think that heterogeneity and variability go hand in hand. The Bortkiewicz theory seems to say just the opposite. This theory has been severely criticized by some theoretical statisticians, but has been for a long time a general practice of insurance companies who seek a wide variety of risks thus improving the steadiness of the financial side of the business.

In the second method of arranging the cards they are grouped into piles each pile having cards with the same or approximately the same observations. Another way of saying this is that the cards are arranged in a frequency distribution. Two general classes of problems arise, according to whether we consider the data to be complete like a census report, or merely a random sample from a larger aggregate. In the first of these general classes we have the problem of describing the data as it is given to us, by means of a few characteristic numbers which may or may not be parameters in some analytic representation which has been chosen. That is, we are simply trying to describe the data that has been given to us without making any general inferences except as we may make allowances for inaccuracies in our observations. For example, the distribution of stature among the students of the university can be closely represented by the terms of a binomial expansion. This part of mathematical statistics is sometimes called descriptive statistics. The questions which arise are those concerned with the choice of the statistical constants used to represent the mass of data, or with the kind of analytic representation which gives a best or most convenient fit, or perhaps with the testing of some hypothesis which has been suggested as to the law or causes of the given distribution. Perhaps a comparison with another group of data is the problem.

If the data given us is to be considered as a random sample from a larger population, a greater variety of problems arises but all go back to the one question, "What are the characteristics of the population from which this sample was taken?" We are now in that part of the field of statistics known as "random sampling" where there is the greatest activity at the present time. There is first, the question of the form of the distribution of the hypothetical popula-

tion, where we have a choice of one of three systematic methods of solution known as the Pearson, or the Charlier, or the Edgeworth scheme. Since a shuffling of the original cards has no effect on the frequency distribution piles, we can say that the constants which are to be used to characterize the distribution are symmetric functions of the observations, and thus many theorems in the theory of symmetric functions can be transformed into statistical theorems.

Out of all the forms of distribution which are at our command there is one which stands out among all others like the straight line in elementary mathematics. This is the so-called normal or Gaussian distribution. Many anthropometric measurements follow this law, particularly stature. Years ago the normal distribution was thought to be wide spread. It was often assumed that if we only had enough data the distribution would eventually be of the normal type. Then later it was considered as a more or less good approximation and now we sometimes see the statement that in practice, if we have enough data we rarely ever find normal distributions. This of course means that as time goes on we are able to recognise finer shades of difference between distributions. Although we may not find many real normal curves in nature nevertheless the distribution is of supreme importance. We cannot dispense with it any more than we can dispense with the right angle because we often hear it stated that the right angle is rarely found in nature.

But the finding of statistical constants which characterize the data is not sufficient. They were found from the information contained in the sample at hand. If we have at our command the results from many samples we can find empirically something about the reliability of the statistic in question. It may be that some other constant would serve our purposes just as well. For example, there are many statistical constants used to measure the amount of deviation from some average value. Two of the best known of these measures are the "mean deviation" and the "standard deviation." The accuracy of the mean deviation is less than that of the standard deviation. In fact the standard deviation calculated from 88 items has the same weight as the mean deviation for 100 items. This makes quite a difference where the cost of collecting data is great.

But we have only one sample. Here we have to fall back on theory to help us. We want something that will save the labor of actual observation. For example, suppose we go out on the campus and select at random ten male students, measure the height of

each one and find the average height. Now suppose this is repeated a thousand times. We have then a thousand observed averages each of which can be written on a card and these cards can be arranged in frequency groups. The frequency distribution will tell us a lot about the dependability of a sample of ten to tell us something about the general question of student stature. The shape of the frequency distribution of averages will be quite different from that of the parent population. If averages of one hundred students were taken instead of ten a still different frequency distribution would be found. The essential characteristics of these distributions can be described by the usual statistical constants, say the arithmetical mean, the standard deviation and the skewness. For example we will find that the standard deviation for the distribution of means of ten observations is about three times that for distributions of means of one hundred observations. The empirical investigation suggested is unsatisfactory and laborious and unnecessary. The whole problem and its solution is summed up in the theorem: Given a normal population with average a and standard deviation s. The distribution of averages of random samples of n variates is normal, having the same mean as the distribution from which the samples were taken and a standard deviation equal to s divided by the square root of n.

I have spoken of the frequency distribution of the averages. Every statistical constant that can be calculated from the sample will have its own distribution and it is the problem of the mathematical statistician to find it. It is rather obvious that the solution will depend upon the distribution of the parent population and upon the size of the sample. The general problem, given a frequency distribution to find the law of distribution of a given statistical constant, has not been solved, but solutions of many special cases have been found, some within the last year. In this part of the problem we see the reason for the recent change in nomenclature from "statistical constant" to the word "statistic." Statistical constants in the field of random sampling are not constants but are frequency distributions.

This problem falls into two merging classes depending upon the size of the sample. The older problem is that of large samples, but it has proved to have little application in many fields where it is impossible to get large samples, as in engineering and agriculture. Within the last few years there has been a remarkable development

of the theory of small samples. Very little of this work has found its way into text books or treatises. It is not a good time to write a treatise on mathematical statistics for the book would probably be out of date in parts before it came off the press.

Each card of our pile may contain more than one piece of information about an individual of the population or the sample. For example, each card may have the height and the weight as well as other information about a student. In this problem we have frequency distributions in more than two dimensions leading to a much richer field of investigation. It takes a model in three dimensions to show the distribution of height and weight. Where in the simple case of classification with regard to one variate we were lead to laws of distribution represented by curves in the plane we now have to consider surfaces in three or more dimensions. We have the problem of characterizing these surfaces by means of statistical constants, and in fact all the problems we have enumerated in a more complicated variety, with the addition of the questions concerning the relationship if any existing between the various sets of data on our cards. This brings us to the wide field of correlation much of which is virgin soil. We really know very little about correlation. When our surface has normal curves for cross-sections by vertical planes and ellipses for horizontal sections or approximations to these we can discuss correlation with a certain amount of confidence, but much remains to be done in the infinite variety of other possible surfaces. The calculation of measures of correlation is an easy problem. It can be done by simply turning the crank of a machine, but the explanation of the meaning of the measure after we have found it is in general a difficult problem and the machine is silent on the subject. However, the correlation coefficient is not the only measurement subject to this criticism. We teachers have often gone through some weird and intricate calculations with regard to students' grades and have come out with, say a report of 77 percent. What do we say when some iconoclastic individual asks, "77 percent of what?"

Running parallel with a discussion of statistics there could be a discussion of another branch of science, the theory of probability, with now and then a connecting bridge. The bridge idea is one conception of the relation between statistics and probability. Another conception is that of a sort of mist through which the statistician must peer. Still another is that the theory of probability is a tool

house for statisticians. We have all sorts of statements from "the theory of probability is dead" to the notion of some that probability and statistics are one and the same thing.

What is the matter with the theory of probability? The answer is a long story and very controversial. In the first place you will find in most of the older text books a definition of probability somewhat as follows: If an event may happen in a ways and fail in b ways and if all of these ways are equally likely, the probability that the event will happen is $a/a+b$. The trouble comes with the phrase "equally likely" which really means equally probable. Thus the definition disobeys one of the rules of a good definition, and arguments based on it come close to the fallacy called "petitio principii" by the older school of logicians or simply begging the question. It is not difficult to understand what is meant by "equally likely" in the problem of throwing a coin or a die, but what does it mean when we say that the probability that a man 40 years old will live one year is .99? A very voluminous literature has grown up about this phrase. The controversy has produced two cults calling themselves "insufficient reasonists" and "cogent reasonists."

The insufficient reasonists say that two things are equally likely if there is no reason to think them otherwise. If the insufficient reasonist knows absolutely nothing about an event he says that the probability that it will happen is $\frac{1}{2}$. This injects the subjective notion into the idea of probability. To an infant the probability that the moon is made of green cheese is $\frac{1}{2}$. To an all-knowing being the probability is either 0 or 1.

The cogent reasonists say that events are equally likely if there be a conclusive reason for thinking them so. A hasty view of the matter would make one think that the cogent reasonists had invented the insufficient reasonist and was using him as a man of straw to pelt with arguments. But too many well known writers have taken the question seriously for us to take the matter lightly. There is a famous paradox used by the cogent reasonists to bewilder the insufficient reasonist. It is usually called the "Life on Mars" paradox.* We ask the insufficient reasonist, "What is the probability of life on Mars?" He answers that he knows nothing at all about the matter, the probability is then $\frac{1}{2}$. Then we ask him, "What is the probability of no men on Mars?" Again the answer

* See *Probability and its Engineering Uses*, by T. C. Fry, pages 117– .

is $\frac{1}{2}$. What is the probability of no dogs on Mars? The answer is $\frac{1}{2}$. What now is the probability that there are neither men nor dogs on Mars? This probability is the probability that two events happen which is the product of the two probabilities or $\frac{1}{4}$. We can continue the questioning to a long list of animals and find the probability that there are no men, no dogs, no horses, no cows and so on to n animals, is $(\frac{1}{2})^n$, and the probability that there is some life on Mars is $1-(\frac{1}{2})^n$ or nearly certainty. Thus insufficient reason gives two answers to the question of the probability of life on Mars.

This is but one example of the many paradoxes which have shaken the foundations of the older theory of probability. A more serious one is the so-called Bertrand's paradox. An example of it is the old needle problem. A needle is thrown at random on a drawing board which is ruled with parallel lines a given distance apart. The needle is shorter than the distance between the lines. The problem asks the probability that the needle when thrown will lie across one of the lines on the board. The solution of the problem usually starts with the location of the position of the needle by giving the co-ordinates of its center and the angle made by the needle and the direction of the lines on the board. The answer is $2l/\pi a$ where l is the length of the needle and a the distance between lines. This solution is the basis for the calculation of π as given in some of the books on mathematical amusements. A needle of known length is thrown a large number of times on a board as described. Count is kept of the number of throws and the number of times the needle falls across a line. Thus we get an approximation to the probability that the event happens. This is equated to $2l/\pi a$ and the resulting equation is solved for π. But if we solve the problem by locating the needle by the co-ordinates of its end points the solution is $l/2a$.

Time is too short to go any further into a discussion of the disturbance caused by these paradoxes, criticisms and controversies. At the present time a reconstruction of the theory of probability is going on. The object is to describe a distinct group of observations in the same way that geometry describes space, or kinetics, motion. In the beginning of such a theory are statements giving the fundamental ideas and axioms which are facts of experience. By deduction, laws and theorems are derived which can be shown to correspond to some observed phenomena. Hence the theory touches experience at both ends of a series of deductive reasonings.

The most discussed reconstruction at the present moment is one

coming from Germany. In this plan a careful definition and discussion of the idea expressed in the word "Kollektiv" is given before the definition of probability. The translation of this word into English has not been fixed, but either population or universe is good, using universe in the logical sense. A "kollektiv" is an infinite repetition process that we can think of and describe and can realize approximately in experience. For example, the repeated throwing of a coin or a die, the drawing of a ball from an urn, the repeated measurements of a physical quantity, the observations on life and death going on in a life insurance office, these are all "kollektivs," or universes.

Each element of the simplest universe has or has not attached to it an attribute. We can mark the presence of the attribute by a 1 and its absence by a 0. Thus in throwing a coin a great many times the universe is symbolized by a succession like 0100111010 $\cdots$ 1's meaning heads, 0's meaning not heads. Such a universe is the simplest of all and goes by the name of an alternative. Given such a universe we then define probability as the limit of the quotient of the number of 1's by the total number of 1's and 0's as the total number increases without limit.

I have only time to point out a few of the differences between the old and this new theory which seems to be gaining ground. The old distinction between a priori and a posteriori probabilities has been thrown out. Every probability is a probability within a universe. The establishment of a universe is the first essential. There is no answer to the question, "What is the probability that Illinois wins the football game to-morrow?" for there is no universe. To have a universe there should be a long series of observations of years when conditions were just as they are this year, each observation bearing the tag 1 or 0 according to whether Illinois beat Ohio or not. The two solutions of the needle problem are both correct. Each solution has back of it a different universe, one found from the sum total of positions of the needle as given by center and angle, the other given by the totality of positions of the two end points. This idea of a universe is not a new one. A familiar problem in calculus which carries a notion somewhat akin to that of the kollektiv is the problem of the mean length of the chords of a circle. For chords equally spaced along the perpendicular diameter of the circle the mean length is $\pi r/2$ while for chords spaced at equal distances along the circumference the result is $4r/\pi$.

To me the chief troubles that will be found in the newer theory will center about the use of the word "limit." In the mathematical definition of limit we have a variable x, a constant a, and a positive number ϵ, which may be as small as we please to make it. If the absolute value of the difference between x and a becomes and remains less than ϵ as x approaches a through any sequence of values, then a is said to be the limit of x. The essence of the definition lies in the words "becomes and remains less." Now in the notion of limit as used in the definition of probability the word "remains" has no place. Any apparent approach to limit may at any time be upset by a long run of happenings or a long run of failures. No matter how many times we may throw a coin and record the relative frequency of heads, there is still a probability that the next hundred throws will all be heads or all tails.

To sum up the problem of mathematical statistics,—it is the reduction of masses of data to a few statistics which present to the mind the salient properties of the data. The problem includes the question of choice of the best set of statistics for the purpose at hand, and most important of all the study of the statistic as a frequency distribution instead of as a mathematical constant. These statistics include the arithmetic mean, the geometric mean, the harmonic mean, the mode, the median and the long list of other averages and index numbers, the measures of dispersion such as mean deviation, quartile deviation, standard error, probable error, measures of skewness, kurtosis and excess, measures of regression and correlation and the unnamed parameters in the various analytical representations of frequency distributions. Throughout all of this we must keep in mind that mathematical manipulation never improves poor data. It usually shows us just how bad the data is. This phase of the problem has been tritely expressed by someone who said, "It is useless to put a razor edge on a hoe." Then, with reference to the theory of probability, we may say that one problem of the mathematical statistician is that of more or less worry over the health of a member of the family.

JOIN THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS!

The Uses of Statistics in the Social Sciences*

By W. L. CRUM
Cambridge, Mass.

STATISTICS AND statistical methods have become so common a tool of the scientist, both for conducting his research and reporting his findings, that the ordinary layman is in danger of forgetting that statistical theory is a branch of mathematical science and that statistical practice is one type of applied mathematics. This important point should be constantly in the mind of everyone who uses statistics, whether as a producer or as a consumer. Nowhere is the danger of ignoring or forgetting this injunction greater than in the various fields of social science. Whether this is due to the great abundance of factual data and their ostensible bearing upon problems of timely interest, to the supposed necessity of putting reasoning in these fields on a quantitative basis at any cost, to the absence of traditional dominance by mathematical thought processes in these fields, or to other reasons, it is idle to speculate. The essential fact is that statistical workers in social science need particularly to be reminded that the vigorous and growing structure of statistical theory takes its root in mathematics.

I shall bear in mind that most of you are actively engaged in teaching mathematics, and therefore are interested in the statistical problems which your pupils may encounter when they come to the study of social sciences. Although my title, "The Uses of Statistics in the Social Sciences," may lead you to expect me to describe at length the specific applications of statistical theory and method in the several social sciences, it is quite impossible to make even a rough attempt at discussing the great variety of statistical analyses recently and currently carried on in social science. What I shall say will come more properly under the head of "The use of statistics," and I shall therefore be talking about generalities rather than about specific problems —about the point of view of, the procedure in, and the limitations upon, the application of statistics to social sciences. Such an approach will not only be more manageable for me, but it will be more

* Read before the Association of Teachers of Mathematics in New England, December 3, 1932.

effective, in that it will give such a thread and unity to what I have to say as would scarcely be possible if I undertook to describe even the chief particular statistical problems encountered in various branches of social science.

We will understand that the term *social science* comprises all scientific thinking which pertains to the behavior of human beings in groups. Thus it has come generally to be regarded as including such particular sciences as anthropology, sociology, economics, history, public law, ethics, psychology, and education. In each of these fields the modern generation of investigators has found it helpful, and even necessary, to use extensive statistical methods in handling particular problems. That the development of the statistical approach has been more vigorous and more successful in some of these fields than in others should occasion no surprise, and that there is a larger element of progress being made in certain of the fields than in others is also to be expected. The situation within a particular field, either because of the existence of the essential statistical data or because of the adaptability of the currently prominent problems to statistical treatment, governs largely the degree to which the science of statistics as it now stands can be used. Perhaps another vital factor should be mentioned: the extent of public interest—not to say popular interest—in quantitative knowledge may be much greater in one field (for example, economics) than in another (as law).

In order to get even a rough working idea of the use of statistics it is helpful to list statistical theory and methods under certain broad heads. For most practical purposes problems of statistical theory and procedure can be classified as those of observation and measurement, those of summarization, those of correlation, and those of sampling. Practically all questions of theory and method fall in these four grand categories. The advantage of considering the categories separately lies in the fact that all the theories and methods falling in any one category to some extent present similar difficulties and raise similar questions.

It will be said that statistical theory can be classified in these same categories whether we are concerned with social science or some other field of scientific inquiry. At least in some respects however the theoretical difficulties encountered in an attempt to apply statistical methods belonging to one of these categories to social sciences are peculiar to social sciences. It is for this reason that we may fairly

hold that, whereas statistical theory is fundamentally a thing by and of itself, it is governed in many details, and in some of its more essential details, by the particular field to which it is applied. This is especially true in most if not all fields of social science—the application of statistical theory in social science brings up an extraordinary number of peculiar problems, the handling of which gives a wholly special aspect to statistical theory as applied to such fields.

Observation and Measurement

Problems of observation and measurement are peculiarly difficult in the various fields of social science, chiefly for the following reasons. A basic obstacle to satisfactory observation is that, in many such problems, the individuals we desire to observe are widely scattered in time and place, and perhaps in some part entirely inaccessible. Frequently the securing of observations depends upon the willingness and coöperation of the individuals observed, and great numbers of social studies of a statistical nature are imperfect or entirely impossible merely because the people to be studied are reluctant and withhold information. For example, in a study aimed at measuring the degree of morbidity due to specified causes, enumerators might expect to encounter all degrees of resistance, from mere carelessness in reply to conscious lying. Nearly all significant social inquiries must be carried out on a large scale, primarily because the lack of homogeneity is so great that a small group of observations is inadequate. Hence, not only is there large play for the reluctance of individuals, but the very size of the inquiry makes the enterprise expensive in time and money. This accounts largely for the important place of government as a producer of social statistics—government can require (to some extent) replies from individuals, and government can also foot the bill for costly surveys. This consideration also accounts for the by-product nature of much of our social data: we get such data as an incident to governmental operations of various sorts. This condition undoubtedly enlarges the supply of certain types of data—but it also gives an unfortunate slant to many bodies of these by-product data.

A second important difficulty is that nearly all social variables are complex rather than simple, and the selection of a satisfactory unit of measurement is accordingly complicated and in some cases quite impossible. For example, if we are studying the sociology of

family life and are seeking data on housing, it is by no means obvious how we should measure the space available for a particular family. The unit to be selected must take account not only of the varying size and constitution of families, but also of the varying character of housing facilities available. This obstacle to satisfactory measurement is so obvious that its mention seems almost trite, but bound up in this complexity of the unit are many of the initial troubles which confront the social investigator. A third—and somewhat related—serious obstacle to social measurement arises from the multiplicity of interrelationships among social variables. This is a condition which has received frequent notice, and is often summed up in the brief assertion that we can not use the laboratory method in social sciences. For example, if we are making a statistical study of the attitude of the American Congress on some issue such as prohibition, it is almost impossible to get a reflection of that attitude free from other factors. Even when the issue comes finally to a head, in the form of a vote on a definite bill, the votes cast—the enumerated statistical data of the problem—are likely to reflect many other factors. It is needless to add that ingenuity can scarcely devise any means of circumventing the difficulty in such a case—truly, the laboratory method is not possible. Too much can not be made of this particular difficulty, because it means that we can almost never secure any social measurement which is not subject to some influence—perhaps a great deal of influence—from other factors in which we are not interested at the time of measurement. It means that nearly all bodies of statistical data in social science show complex variability due to a variety of factors, only one—or at most few—of which we are likely to be studying in a particular case.

A fourth fundamental obstacle to social measurement appears in the fact that many statistical studies in social sciences are concerned with changes over time, and such changes occur in a more troublesome way in social science than in physical science or natural science. Perhaps it is not so much a matter of speed, it is rather a matter of caprice and irregularity in the movements over time. In any case nearly every branch of social science contains standard examples of statistical time variations, wherein the measurement problems are peculiarly troublesome. A special aspect of this time variation matter appears in the fact that the very definition of social variables, as well as the unit of measurement, changes with

passing time. For example, in the field of political science, the very definition of one of the great political parties in the United States—largely without conscious effort on the part of its members—has undergone important changes in the course of a few generations, and any statistical measurements relating to such an entity are somewhat impaired by this inherent variability in the entity itself.

Quite likely other important obstacles to accurate measurement in social sciences could helpfully be listed, but I will be content to add that the very fact of our dealing with human beings seems to be at the root of most of these difficulties. This accounts largely for the complexity and variability of the objects of measurement, and also for the difficulties in the way of securing measurements. In the eyes of the social statistician it has truly been said, man is not only diverse, he is assuredly perverse.

We ask now about the status of existing observations and measurements in these sciences. We have, to be sure, a large and growing body of statistical data in various fields of social science. Back of this body of material is the disquieting fact that all observations of this sort must have been made in the face of the many difficulties discussed above. It is therefore impossible to accept the existing body of statistical information as a basis for analysis and interpretation without first inquiring into the specific manner of making such measurements. It is by no means my purpose to suggest that we should take existing data at their face value in other sciences; but I insist, in the light of what I have said above, that we must be peculiarly skeptical about the adequacy of existing statistics for any given investigation in social sciences. It is especially important, before proceeding to the use of such statistics, to inquire into all of the circumstances surrounding their collection and compilation. It is hardly necessary to remark that many of the nonsensical published inferences drawn from census tabulations, vital statistics of particular communities, and numerous other bodies of social data, can be traced to the failure of the interpreter in this respect—he failed to take the essential step of informing himself about the raw material which he proposed to use.

Before leaving this aspect of our subject, it may be well to remark that the case is by no means hopeless. Important improvement in the quality and adequacy of social statistics is not only possible, but confidently to be expected. Some of the difficulties we have

mentioned above are fundamental and unalterable, but they do not completely prevent our eliminating some of the worst defects which affect the existing body of social statistics. Some improvement can be made in existing statistics, by proper selection and qualification; but much greater improvement can be made in data hereafter to be collected. Once the user of such statistics frankly and clearly recognizes the limitations and the reasons therefor, he has immediately an incentive for inventing means of improvement; intelligent and discerning use of bad statistics is the surest way to secure better statistics.

Summarization

If I have dwelt at some length upon the problems of observation and measurement, it is partly because the machinery for summarizing and interpreting statistical data has been developed much more fully than the means of measurement (at least in most social sciences), and also because the greater opportunities for improving our statistical knowledge in social science now appear to lie in the elaboration of schemes of measurement rather than in the extension of the machinery of analysis. It is nevertheless true that the machinery of analysis made available by statistical theory must be specially adapted to the problems of social science, and it must be used in such problems with a full realization of its special limitations.

The summarization of statistical data is usually accomplished by the calculation of averages and ratios and other derived statistical characteristics; and, in the ultimate case, by fitting a curve and deriving its mathematical formula. The carrying out of these operations in a problem of social statistics is likely to uncover important obstacles and limitations. Broadly speaking, the statistical distributions typical of the social field do not have the simple and highly uniform shape which has furnished a guide in the development of statistical theory. History makes clear that statistical theory takes its origin from the mathematical study of games of chance; and, with some qualifications, we discover also that the particular games of chance which were initially studied were of a remarkably simple sort. The result has been that the very definitions of some of our simplest statistical concepts, such as the arithmetic average, reflect the properties of simple and highly uniform frequency distributions such as arise in these simpler games of chance.

The study of modern masses of statistical data accumulated in

the various sciences (not only social sciences, but also other fields) has brought to light great numbers of statistical distributions which depart notably from this elementary type. These departures are peculiarly frequent and peculiarly marked in social statistics, and in every problem of social statistics our first step in the analysis of the data should be an examination of the form of distribution, in order to discover how far removed it is from the elementary type fundamental to statistical theory. For example, in the field of economics we are frequently interested in the quantitative study of wage rates. Unless the group of workers for whom our wage data are compiled is extraordinarily homogeneous, the frequency distribution is likely to have more than one mode or maximum point; and such a distribution is not properly susceptible of summarization by means of the arithmetic average and similar statistical characteristics.

This basic difficulty, the lack of simplicity or smoothness in the statistical distribution, is the controlling factor in the analysis of most social data. This frequently means that we have to devise, for a particular statistical problem in social science, a set of special summarization methods. It generally means that in such work, we must place reservations upon the indicated results obtained by following customary practices and using customary statistical characteristics. It almost always means that our process of summarization must stop short of the final operations—fitting a curve and deriving its mathematical formula—and I am tempted to say that the presentation of a mathematical formula as the end result in a statistical analysis in social science ought to be taken by the reader as presumptive evidence that the investigator has tried to prove too much.

Since so many statistical inquiries relate to changes over time, as mentioned above, there has been developed an entire system of summarization devices peculiarly designed for the interpretation of time series. This tendency has been greatly aided by the abnormal emphasis placed in recent years upon the study of the business cycle and all related social phenomena. The lively public interest—sometimes verging on the sensational—in business cycles has on the one hand stimulated an undue growth in this part of our science and on the other hand made statistical specialists unpleasantly open to attack because of their blunders in the treatment of a tremendously complicated problem. At any rate, we have this considerable body

of peculiar statistical methods adapted to the study of time series, many of which have been worked out by amateurish investigators who are either ignorant or disdainful of the theoretical foundations of our science. The analysis of time series may perhaps be reckoned an exception to the general statement above that the progress of theory has outstripped the improvement of underlying data. It is perhaps better to say that in the analysis of time series we now have such abundant data and such good data that we ought to place greater effort on the theoretical criticism of our methods of analysis. I will not stress this point, because it is still true that vast room for improvement exists in the gathering of basic data.

Correlation

If we were to single out the one element in the structure of statistical theory which has received outstanding attention, both from the point of view of its development and its use, in recent decades, it would have to be *correlation*. Correlation is not a twentieth century concept, but had already a fairly considerable history when, in the nineties, striking contributions by Professor Karl Pearson brought it emphatically to the attention of the scientific world. Its attractive formula, with its smoothness and symmetry, facilitated its rapid adoption in practical work, and its exceptional capacity of apparently producing something out of nothing made its appeal to the modern superficial statistical investigator practically irresistible. Amplifications of the fundamental theory of correlation have on the one hand enlarged the scope of its applicability, and on the other hand strengthened the weaker elements in its formulation. There have been so many improvements in the technical devices for applying the method in specific problems, that correlation, although it has by no means become fool proof, has certainly been made available for use by the foolish. Without the slightest hesitation I say that the method of correlation is one of the most dangerous tools which modern science has placed in the hands of social investigators. I am not overlooking the enormous potentialities of the method, as an instrument for producing worthwhile results. I am merely trying to lay before you emphatically the view that correlation, because of the extreme simplicity with which it can be applied and the ostensible finality of its indicated results, is likely to be used—and has

frequently been used—to produce grossly misleading and sometimes basically erroneous interpretations of statistical data.

The dangers incident to the application of correlation methods are especially great in the social sciences for the reason, noted above, that in most problems of social science the number of variables involved is large and the interrelations are complex and not subject to representation by simple mathematical formulas. Accordingly, the statistical investigator in these sciences who undertakes to apply correlation methods is quite likely to think himself impelled to use the more complicated methods of partial and multiple curvilinear correlation. These methods are especially insidious, and it is almost impossible, even for the mind trained in higher mathematics, to obtain a mental image of the compound frequency distributions involved, and the implied interrelationships among the numerous variables. The statistician who embarks upon a journey of inquiry with a multiple correlation coefficient (or its associated multiple regression equation) as his sole guide, presently finds himself storm-tossed, without rudder or anchor, and compelled to rely upon no force except that of plain chance. The striking fact is that these methods are most widely used by investigators who are blind to the theoretical difficulties, and so proceed happily unaware of the dangers which surround them. I shall be regarded as the veriest mossback, and yet I will say that if I had to give but one bit of advice to those who may sometime do statistical work it would be that they shun multiple correlation as they would shun the plague.

Nevertheless, it is fair to notice the very real power which we have in this extraordinary analytical device: wise use of the method of correlation, confined to problems for which it is appropriate, may very well yield immensely valuable results. As is so often asserted, the method gives a substitute—poor indeed, and only a makeshift—for the laboratory procedure which "keeps other things equal." I hesitate to state the case in this way, because careful examination of the partial correlation coefficient makes it perfectly clear that this device does not keep all but one of the independent variables constant. Through frequent and persistent carelessness this idea has become current in the literature, and these partial coefficients are quite generally cited as measuring the degree of correlation between two variables while others are kept constant—this is quite erroneous.

Sampling

The fourth category of theoretical problems is sampling. There would never be any problem of sampling if investigators did not seek to generalize from calculated results, based upon study of a given group of individuals, to statements concerning a larger and more inclusive group of individuals. Cases in which the desire for such generalization does not exist are few however, and we may risk the general assertion that nearly all problems—particularly in social fields—aim ultimately at passing from the results of a sample to general conclusions about a larger population. Hence the use of samples is almost inevitable in statistical inquiry in the social sciences, and a theory of sampling becomes an important adjunct of any systematic use of statistical methods.

At this point we must take up again the history of statistical theory, and note that the sample about which statistical theory talks is of a very special kind. Once more, the kind of sample had in mind in the development of statistical theory is such as arises in connection with fairly simple games of chance. Even though such a sample may not be concerned with a game of chance, the manner of its construction must be such as lends to definition in terms of games of chance. Thus it turns out that, when we try to construct such a sample in a problem, we must generally adopt methods of chance. For example, if we desire to learn the average weight of the adult male citizens of a nation, we shall undertake to measure only a limited number of individuals—a sample. If we are going to have this sample amenable to the standard theory of sampling, it must be a random sample, and one of the ways of accomplishing this is to select the individuals of our sample from the entire group by lot,—that is, by chance. Now I call you to witness that most actual sampling in the fields of social science is not done in this way: instead of selecting samples by chance and with corresponding regard for the rules of random sampling, investigators, in their self-reliance, undertake to construct what they call *representative* samples. These hand-picked samples can in particular cases be extraordinarily good representatives, but the point I want to emphasize is that the use of such arbitrarily chosen samples is at the expense of renouncing whatever benefits the theory of sampling confers. The theory of sampling gives us certain extremely valuable information about the rep-

resentativeness of our sample, in case it is a random sample, or in case it departs moderately, and in a known way, from the random.

This theory has, as one of its end results, so-called probable errors. The probable error of a particular calculated statistical characteristic gives us an estimate of the precision of that calculated result as a measure of the corresponding unknown magnitude truly pertinent to the entire population. Thus, in connection with our average weight problem mentioned above, we would get from our random sample—notice it is a *random* sample—not only an average weight of the men in the sample, but also a probable error of that average. Having this probable error before us, and granting that certain other requirements are met, we should be able to say that there is an even chance that the average weight of the men in our sample does not differ by more than the probable error from the average weight in the whole population. The significance of an achievement of this sort is so great that, if we were talking about any field of human interest except a science, it would be regarded as truly sensational. Here indeed, out of our ignorance we gain knowledge—by choosing a sample in the way which leaves the least room for play of our supposed intelligence, we get a result with reference to which the theory enables us to state the probable degree of accuracy for purposes of generalization. This magnificent achievement of mathematical theory is rudely cast aside by unnumbered amateurs who call themselves statisticians and profess to be able to select, by following their unaided intellect, samples to which they impertinently attach the name *representative.*

I know of no field of science in which it is more desirable to get back to the method of random sampling, which statistical theory teaches us, than the social sciences. One reason is that in the social sciences investigators are peculiarly in danger of being swayed by subjective considerations. The issues involved are so complex and difficult to understand, and they affect so closely the lives and fortunes of great numbers of people, that it is almost impossible for an investigator to maintain a dispassionate and wholly objective point of view. This applies to our very best investigators who have the very highest devotion to the scientific point of view; and, of course, the ordinary superficial student of statistics is only slightly protected by this armor of scientific integrity. It seems to be clear that we should make a special effort in statistical work in social

sciences to avail of those methods which leave the least play for subjective influence. This strongly points to the systematic use of random samples in the study of these problems.

Conclusion

We have made a hurried survey of the way in which statistical theory and method must be specialized in applications to social science. We have emphasized the basic mathematical nature of statistical theory, and have seen that sound use of statistical procedure in social science depends upon constant testing in the light of simple mathematical fundamentals. Great strides have been made in the reduction of the ideas and doctrines of social science to quantitative form through the medium of statistical analysis, and further great strides are in prospect. But I have quite wasted your time if I have failed to make clear that all progress of this sort is conditioned upon wise use of an intellectual machinery which, because of its inherent mathematical nature, can tolerate neither inconsistency nor carelessness.

Mathematics — A Mode of Thought

Mathematics is not, as is sometimes imagined, a distinct science. It is rather a mode of thought and a form of expression, precise in quality and universal in application. Ignorance of a foreign language or of any single branch of science may be excused. Ignorance of mathematics undermines all positive thinking. Engineering practice, actuarial theory, the principles of sound banking and safe corporate organization, the fundamental laws of economics are not capable of expression except in mathematical terms. As society progresses and social complexity increases, a general comprehension of mathematical principles is well-nigh indispensable. With the understanding of mathematics comes a correct interpretation of the social significance of taxation, insurance, financial security, depreciation, capitalization, and distribution.

The high school is an essential factor in universal popular education. Mathematics, next to English speech, is its most useful medium of expression.

Alice in Numberland

By Blanche B. Hedges
The Gardner School, New York City

"What is 4 raised to the 4th power?"

"16," answered Alice, sweet and low.

"16? Even those baby freshmen know better than that. Off with her head," exclaimed the Duchess, as she raised her eyebrows in a parabolic curve.

"She is mad as the March Hare," whispered MaBelle Starr.

"Who?" replied Constance Variable, a guardedly wandering eye looking volumes.

Just then a gust of wind blew in through the window. It was not a gentle, perfumed breeze wafted through the bloomin' pear tree, but a moaning, haunting sound like the wind groaning in dark cypress trees in a dismal swamp. It was a living, vital Thing bringing its miasmal breath in to the hitherto bright and cheerful room. The Duchess seemed unconscious of the sound, or of the Presence; but the girls, in spite of the strict attention they were giving to their lesson, drew closer together for protection from the Unseen Thing.

"4 raised to the 4th power equals 16? Show the Why," droned the Duchess.

"Because the square on the hypotenuse is very equal to the squares of the embracing arms," murmured Alice, dreamily.

"Ummmmmm," grimly snorted the Duchess. "I seeee. I seeee which way the wind is blowing. So that is the Why. Attention young ladies, no giggles! Away with sentimental nonsense! The implications in the student's answer are entirely too obvious for us to ignore. I can draw only ONE conclusion from such a premise. What is yours?"

"Oh, have a heart," recited the class in unison.

"I may, indeed. But now, young ladies, let us consider the question academically. Her Syllogism—"

Again came that shivering, intangible Presence as though the word syllogism had been the abracadabra for its summons. With it a stronger gust of wind blew through the open window, caught up the loose lesson papers lying on the desks, and scattered them like stray thoughts all over the room, blowing them around in perfect circles,

or eddying curves—happily unsolvable mathematically. It was as though some Unseen Thing had entered the room stealthily, and with malice prepense, bent on mixing things up generally.

"Oh! there it is again, Duchess," wailed the frightened girls, and they all huddled together, drawing closer to her protecting arm—that strong arm, ever ready to wield the chalk in a lost cause. There the little group stood, trembling and shaking with Fear of they Knew not What: their eyes were wide with apprehension and dismay: on their faces, lovely even in this appalling and fearful moment, were expressions of anxiety, dread, terror.

"All right, girls," bravely chattered the Duchess, between her chattering teeth. "Whatever this Horrid Thing is, that haunts us, and makes us start like guilty things upon a guilty summons, you and I will fight together for our honor and our lives. One for all! All for one! Rah! Rah! Rah! Princeton!"

Alice raised her hand—still quaking. "Oh Duchess! Last night I dreamed about the Abominable Thing. May I tell my dream?"

"Yes," said the Duchess—anything to gain time till the rescuing dismissal bell.

"I thought I was sitting on a very high ricketty fence ready to tumble either way," began Alice dramatically. "Before me all was black, empty, abysmal. Behind me was the lovely crescent of the Baby Moon, formed by the intersection of two circles of unequal radii. As I looked at it over my left shoulder, by the principle of continuity it changed into the perfect circle, the area of which even in my dream I knew to be Πr^2. As I revelled in its intellectual beauties suddenly the warm, still, summer air became damp and intellectually chill, and I grew clammy with the cold. A THING—of appalling weight seemed to be sitting on my wrong shoulder. It was whispering horridly in my ear that slogan of the Great War—'Ils ne passeront pas.' Horrors, said I to myself. It speaks FRENCH. IT haunts the Math room. WHAT is IT? Bravely I turned to rout the Beastly Thing by the psychological principle of substituting a happier thought, and muttered reproachfully, 'Et tu, Brute.' It gave me a wise and wicked wink. I nearly fell off the fence—IT knew Latin, too."

Here Alice was so overcome by the horror of her discovery, that she stopped to fill her fountain pen, and then continued shudderingly.

"Hamlet's immortal words came to my help—blessings on the English department—and I addressed the Foul Thing.

" 'Angels and Ministers of Grace defend us.
Be thou a spirit of health or goblin damned,
Bring with thee airs from Heaven or blasts from Hell,
Be thy intents wicked or charitable,
Thou comest in such a *questionable* shape
That *I will not have speech* with thee.
Thou wouldst deprive me of the sovereignty of my reason
And draw me surely unto madness.' "

By this time the poor girl was in an almost fainting condition, but some one thoughtfully brought a handy literal, and she was enabled to continue her narrative with greater strength and confidence.

"Unable to bear more, I would have fallen into the abysmal depths behind me, had not the strong arm of a handy derrick reached out to hold me in my precarious and unsafe position on my tottering fence —the derrick arm operating through an angle of 45 degrees, and under the formula $P=\frac{w}{n}$. I sobbed, 'Oh save me,' and snuggled up close to that iron arm. And then mirabile dictu—as though my touch was like the kiss of the Prince on the lips of the Sleeping Princess, that cold arm became a soft, warm, cozy one. Duchess! do you know the answer to my dream—you who solve so many problems? By gad! I'll give you 100%, if you tell me the correct solution to this one."

"Let me think," cooed the Duchess, spinster but sporting, so thrilled by the new problem that she ignored the naughty word of the naughty Alice. "For one thing—you will 'get by' in your English. That is excellent. But there are also so many mathematical principles involved, I shall have to see if I can reduce them to lowest terms for your intelligence. The class may do Home Work while I ponder."

But even then in the quiet of that sunny room around the innocent, curly heads of those tender maidens hovered that Malign Presence, as they plied their pencils busily in the mad preparation of the morrow's work. Even giggles and surreptitious tickles failed to banish it. THE NAMELESS DREAD WAS THERE.

"Attention, girls, I have a thought," said the Duchess, interrupting their perfect concentration. "I seem to see the long, oblique rays of the summer sun shining through the Gothic windows over a high altar. By the way, Alice, how would you determine the height of that altar?"

"By trigonometry," glibly answered Alice. "By using the function $\text{Tan } A=\frac{a}{b}$."

"Right-o," smiled the Duchess clapping her hands at a correct answer. "The bright light coming through the colored glass casts the warm glow of the crimson and of the blue of Harvard and of Yale upon a handsome couplet waiting to be made one. In spite of the brilliant spectacle and the swelling notes of the rich-toned organ, there is a sense of apprehension among the bridal party, and even in the large and fashionable congregation. This Nameless Dread is particularly apparent in the piteously anxious expression on the face of the lovely and dimpled bride. All feel THE HIDEOUS PRESENCE, still hovering about, its dark wings casting shadows over all. Is there no Magic that can vanquish IT?"

"As the Shining Youth places the Ring—the mathematical symbol of Eternity (circumference equalling $2\Pi r$) upon the waiting and eager 4th digit of the trembling maiden, I hear the solemn words 'I take thee Alice, to be my wedded wife—for richer, for poorer, for better, for worse.'

"Then Alice, not in the low, whispered, uncertain tones of the classroom but in a clear, triumphant, bell like voice responds:

"Love is Magic. 4-4-4-4—are 16, after all. I was right; Duchess was wrong, in insisting on multiplying, in those good old days. Oh my Sir Knight! I shall truly be unto thee as 16 wives—4 richer, 4 poorer, 4 better, 4 worse. I will unite all their virtues in my one small person."

"Then with a flash and ringing clang, 256 gallant ushers from the Point and the Academy draw from their scabbards their glittering swords, and as 256 St. Georges, enough for the whole school, stand forth with broadswords raised high—even to the nth power, ready to destroy all enemy dragons that threaten the peace of mind of pretty girls."

"As the clank of armoured steel grows faint in the still church it ends in a long drawn out sound like a cross between a snort of baffled rage and a sad sigh. The little bride in the safe haven of her husband's arm sees at her feet on the altar step that DREAD, HAUNTING CREATURE of scholastic years. College Entrance Examinations—laid low by Righteous Murder and Sudden Death—a Sacrifice to LOVE."

EPILOGUE

And they lived happily ever after.

Never did the Ugly Ghost rise again to haunt that Happy Home.

For the Heavenly Twins, Eleanore and Sylvia, passed successfully, always with Honor and with Joy through a CERTAIN SCHOOL, where the Dread Bugaboo speaketh no more in Latin, or in French, or in Math, but has been banished forevermore.

Quoth the Bluebird—"Forevermore."

Mathematics for Everybody

Outside of their own immediate occupations, the millions of our people can seldom make use of any mathematics beyond counting, adding, subtracting, multiplying and dividing. Yet 60,000,000 persons own stocks—other millions buy on the deferred payment or installment plan; everyone buys, nearly everyone sells, something. Many plan and construct. No one escapes taxes, debts, wages, salaries, profits, losses. Judgments have to be formed, decisions must be rendered on all of these things. But no sound judgment and no intelligent decision can be rendered in these matters on the basis of a counting and adding mathematics.

After men have counted and measured they must seek, and they often find, the relation or relations between the quantities counted or measured. Then they try to describe these relations, and finally to express them as formulas. When this is done men are able to predict the outcomes of quantitative situations. This procedure is mathematical. This type of thinking is practiced only in the mathematics classroom and it must be offered to all.

Men starve in a land of plenty because the mathematical efficiency of production is denied to distribution. When our social, political and financial leaders saw the impending crash in 1928–1929 the millions who could not think mathematically overwhelmed them. Education for an improved, happier social order must continue to provide every child with a proper and adequate mathematical training.

Daily Testing

By WALTER H. BLAISDELL
New Britain, Connecticut

INTRODUCTION

THE CONTENT MATERIAL of this article is intended to demonstrate that daily testing can be made to serve several purposes, i.e., to educate the pupil not to waste his time, to induce him to evaluate his past performance and to develop original thought. The experiment which has furnished the material for this article was and is being performed with various ninth grade algebra classes, has thus far covered somewhat more than four years in time and is, as yet, far from being complete. These classes consist, for the most part, of pupils of foreign parentage, having low reading quotients in comparison with their intelligence quotients. The outstanding characteristics of such pupils indicate that they do not know how to use their time to good advantage, but that they show a decidedly favorable response to the stimulations offered in the procedure outlined below.

PEDAGOGICAL PROCEDURE

Before a class enters the classroom, the instructor has copied a short test of five questions on the blackboard. (The number "five" is an arbitrary one and is so chosen for the sake of convenience.) Experience has shown that, if a group of pupils knows that such a test will be ready and waiting for it, the members of that group will tend to arrive earlier than they ordinarily would. The first arrival finds some paper, usually scrap or waste paper, waiting for him. He quickly distributes this at several strategic points about the room. Other pupils, on entering, pick up a piece of paper on their way to their seats, and proceed with the test. It seems worthwhile to state here that it is rare for a pupil to come to class without a pencil, a habit so common to the boy or girl of ninth grade age. The reason seems to be that the pupil knows his friends are going to be too busy to stop to lend a pencil. A time limit is set for this test, which usually culminates one or two minutes after the opening bell of the period. The first one or two pupils to complete their

tests take it upon themselves to collect the tests of the remainder of the class. After the tests are collected, the instructor conducts an explanation and solution of the questions as seems advisable, sometimes asking members of the class to do this orally, and in any event, having the solution and answer written on the blackboard. At this time the collected tests are distributed among a few competent pupils for correction. After being corrected, tests are passed to another pupil, who scores them on a record card maintained for that purpose. At the end of a week or similar convenient period, daily scores are totaled, the percentage computed and graphed by the teacher on a sheet of coordinate paper placed for the graphing of various test scores in the text book of the individual. Comparison of score graphs by class members emphasizes to each pupil his own accomplishment.

The time required for this procedure seldom exceeds ten or twelve minutes, usually about three minutes needed for the completion of the test, the remainder being spent in class discussion of the questions. While the discussion of the questions is in progress, pupils are encouraged to copy test material which they did not understand in a "model book." They are encouraged not to include in such model books material which can be found elsewhere. The individual pupil is given the responsibility for including that material which he can use later, and he soon learns the importance of such material upon his subsequent achievements.

A series of five such tests, representing the testing material of a one-week period, is introduced here to illustrate that which follows.

9" Algebra, Second Week, Friday

1. $A=p+prt$; $r=$? (in terms of A, p and t)
2. *If x is 3 less than y, make an equation, showing the relationship.
3. Divide 42 algebraically into two parts.
4. *Represent a fraction whose numerator is one third of the denominator.
5. $\frac{a+b}{b}=-1 \quad b=$.....?

9" Algebra, Third Week, Monday

1. *If $.05(1000-x)$ is 7 more than $.04x$, make an equation, showing the relationship.
2. If $.45x=90$, $x=$.....?
3. *If $5(20-x)$ cents and $25x$ cents have a total value of \$3.40, make an equation, showing the relationship.
4. *If the rate percent is a, and the principal is x, the interest is .....?

5. If the numerator of a fraction is 15, and the denominator is 20, the fraction is .....?

9" Algebra, Third Week, Tuesday

1. *A truck leaves the town A for the Town B two hours ahead of a passenger car but is overtaken two hours later by the passenger car. The truck travels .....? hours. The passenger car travels .....? hours.
2. Check this product of two polynomials by substitution,

Let $x=2$. $\dfrac{3x^2+x-2}{2x-3}$ $\dfrac{}{6x^3-7x^2+7x+6}$.

3. *If $x+3y=9$
$x+2y=7$ $y=$.....?
4. x dimes equals .....? half dollars.
5. If $3x-y=7$, and $y=1, 3, -1, -3$
$x=$.....?, ...?, ...?, ...?.

9" Algebra, Third Week, Wednesday

1. *If $3x+5y=19$
$3x+3y=15$, $y=$.....?
2. If $a=1$, $b=3$, $c=2$, $(a-b-c)^2=$.....?
3. The square of $-3ab$ equals .....?
4. $\dfrac{n+1}{4n}-\dfrac{n+2}{2n+4}=$.....?
5. $\frac{1}{2}X=\frac{1}{4}(12-X)-\frac{1}{3}(13-X)$, $X=$.....?

9" Algebra, Third Week, Thursday

1. *If $3x-y=3$
$3x+4y=18$, $y=$.....?
2. If a train travels $2x+18$ miles from Town A to Town B, and then travels $5x$ miles from Town B to Town A, make an equation showing the relationship.
3. A boat travels at $(a+b)$ m.p.h., covering $(c-d)$ miles in .....? hours.
4. The first step in solving problems is .....?
5. *If a quart of water is added to a mixture of 1 qt. water and 1 qt. alcohol, how is the amount of alcohol affected?

For the benefit of the reader, the designation at the beginning of each test was substituted for the date in order that parts of the tests may be used in connection with later classes. (Graduations in this school occur twice each year.) Those questions marked thus (*) indicate that the material covered by the questions so designated has not been previously studied by the pupil.

Selection of the Question

The reader can understand that a large part of the profit that the pupil derives from such testing depends upon the care with which the questions are selected. For convenience in correcting

and graphing, these general rules should be followed: (1) The question must be of simple construction, involving no ambiguity of question or answer. (2) The question must deal with fundamental subject matter. (3) The question must be chosen to meet a definite need, either to establish a principle or illustrate a process.

Those questions comprising the test may be divided roughly into two general classes, those covering work that the pupil has studied and with which he is expected to be familiar, also those which might be termed "thought-provoking," involving subject matter that is to come. The first class needs but little mention except that the instructor has here an excellent opportunity to present formerly studied topics from new points of view. As regards the "thought-provoking" type, a good deal may be accomplished by the instructor who cares to take the time to prepare a series of the same type of example arranged in a sequence of increasing difficulty. If one of these examples is presented in the daily test on each of a number of consecutive or intermittent days, by the time that the class is ready for the more formal presentation of this type of new work, some of the class members are ready to proceed without instruction, and all have some conception of the new methodology. Thus, using a test of five questions, several series such as described above may be operated concurrently.

Simultaneous equations offer a typical illustration of such a series and the method of its presentation. (See model tests.) The first of this series was presented on Tuesday of the third week of the semester. More than fifty percent of those taking this test solved question No. 3 correctly. As can be seen, the tests of the next two days included similar questions, each being somewhat more complex than the previous one. Inquiries divulged the information that the majority of the first few questions of this sort were solved by inspection. Some pupils, however, were quick to pick up the conventional methods. When the author wrote the answer to question No. 1 for Thursday's test, he made an intentional error, writing $3x=21$, $x=7$. An emphatic protest arose, and from several members of the class, indicating an active interest as well as knowledge of the correct procedure. Several questions of this series, following those already given, are included here:

Fourth Week, Monday: $\begin{cases} 5x+3y=14 \\ x=y \end{cases}$

Fourth Week, Tuesday: $\begin{cases} 5x-6y=11 \\ x=y \end{cases}$

Fourth Week, Thursday: $\begin{cases} 3x+4y=15 \\ x+4=y \end{cases}$

Fourth Week, Friday: $\begin{cases} 5x+3y=1 \\ x+y=-1 \end{cases}$

Fifth Week, Monday: $\begin{cases} 5a-b=17 \\ b=\dfrac{1-a}{a} \end{cases}$

Pupils were not instructed as to which methods to use in these problems. In fact, they did not hear mention of any method until the simultaneous equations were presented as a class exercise, at the beginning of the sixth week. Being curious as to the methods that were used, the author conducted a personal investigation from day to day, examining test papers and asking questions when it could be done without interfering with the normal work of the day. Inspection, addition and subtraction, and substitution were common methods. In one case, an "A" pupil disregarded the easier methods in example No. 3, Third Week, Tuesday, but found the correct root by the comparison method, using the longer route of solving for "x," and finding the required value for "y" by substitution. The cardinal principle of making use of the subject of algebra to develop original thought seems to be well illustrated in this extended and possibly pardonable comment.

The author once had the idea that, if he copied a set of seemingly appropriate questions each day for a year, the task of composing such tests would be over. That has not proved to be the case. The most satisfactory method has seemed to be to keep a sheet of paper nearby as the daily homework is being corrected and to jot down items that seem to be stumbling blocks to those that cannot think or that do not wish to do so. This applies, of course to the "review" type of question. A knowledge of the textbook is an advantage in looking ahead a week or two to find new material that will be both pertinent and stimulating. There is but little advantage in including a question that is failed by a whole class, and when one of this sort is found, it should be discarded.

Composing the test as one corrects the home work of the day enables one to take the pulse of each class, figuratively speaking, and to diagnose its ailments, using the daily tests for the cure.

Utilization of the Results

The results of the tests may be found useful to both teacher and pupil. Inasmuch as the difficulty of the test is graded to the intelligence and progress of the class, the pupil may justly compliment himself if he makes a good score, and he has the opportunity and incentive to correct his faults if he makes a poor one. His test graphs are an indication of his accomplishment from day to day and from week to week. He quickly learns that those questions that deal with unfamiliar material were chosen to help him in his future difficulties, and that it will profit him to keep in touch with the material in his model book. No attempt is made by the author to check on model books and the included material. The books are so universally accepted as a pupil aid rather than a retarding influence that the instructor does but little except to suggest methods as to the more efficient ways that such a book may be utilized. At the end of the first half year, even this becomes virtually unnecessary.

From the teacher's point of view, the graphs of the weekly score of each pupil may be used as a check on his accomplishment, although the results as such should be accepted with a degree of reserve. As the questions are heterogenous in nature and as there is a time limit set for the completion of the test, some credit for mental agility instead of knowledge of the subject should be given to the pupil.

It seems reasonable to suppose that this agility is accentuated as the class progresses, also that habits of quick analysis are developed, when it is understood that, in the course of a year's work in this subject, the pupil considers approximately nine hundred questions, each chosen for the express purpose of inducing the pupil to think. This figure is contrasted with the approximate two hundred and fifty questions included in the total number of final tests (ten), distributed at fortnightly intervals.

Weekly percentages range from a minimum of 20% to a maximimum of 95%, although percentages in the immediate vicinity of either of these limits are rare. The author gets the impression that those pupils destined to receive grades of "A" or "B" tend to have but little variation in the percentages of their test scores. The lower the ability and industry of the pupil, the greater is the variance from week to week. A lack of data prevents this from being

more than a theory, but the cause of the variance would seem to be more a matter of irregular study habits rather than lack of ability.

It has been the author's opinion and experience that he has developed the daily testing program to such an extent that it is possible to spend much less time in class discussion and demonstration than was done formerly, thus permitting more work with the individual. There are certain units of work that defy, thus far, the formation of a satisfactory approach by this method. Polynomial division, square roots in arithmetic, and simplification of fractional surds are good illustrations. Such instances are rare.

The pupil attitude, from the beginning, has been favorable. It does not require a great deal of time for him to learn that the project is for his advantage, inasmuch as he is rated on what he learns and not on what he does. Without such approval, any method of pedagogy must be severely handicapped.

It seems obvious that a pupil studying any cumulative subject such as mathematics and held to periodical requirements is subjected to a strong tendency to concentrate his attention on the current topic and consequently to neglect periodical reviewing. It is particularly the function of the secondary school mathematics teacher to unify and coordinate material of the past, present and future, insofar as the sundry topics of these periods are related to each other.

Speaking in a figurative sense, we may consider each unit of work, as fractional equations or rearrangement of formulas, as a high brick wall stretching across the path of the student. We may liken the vertical face that first presents itself to the traveller as the oftentimes abrupt, unannounced and concentrated discussion or explanation concerning the unit of work as presented by the teacher and text book. Supposing the pupil scales the wall, as he often does not, makes his way across the top (through the body of the unit) and descends on the other side. That his descent may not be too abrupt, modern pedagogy prescribes an incline in the form of reviews, tests and sundry forms of remedial work.

The better type of student makes the ascent without difficulty, principally because he has the talent of sifting the pertinent mathematical facts from his collection of acquired information. For the mass, travelling the educational highway, there seems no reason for withholding from the pupil traveler the privilege of the

use of an incline in the *ascent* of the wall as well as in the *descent*, the former being naturally more difficult than the latter. There are various ways in which such an approach may be supplied, daily testing lending itself very handily to such a project.

Conclusion

Those advantages described in this article that seem to be outstanding in pertinence are:

1. The pupil learns to save time, (a) by getting to work immediately on his arrival in the classroom and (b) by including in the equipment that he brings to class those things that he is reasonably certain to use.
2. The pupil keeps a "model book," from which the benefits are directly proportional to the care with which it has been maintained.
3. The pupil is stimulated to think for himself, the thought having a direct bearing on the grades that he gets and upon his general mathematical knowledge.
4. To a certain extent the testing program is a self-teaching device, eliminating a good deal of class demonstration, unnecessary for the upper half of the class.
5. The pupil is enabled and encouraged to grade himself in accomplishment, regardless of other compensations.
6. The opportunities offered to the pupil in the way of quick analysis and selected reviews add much to his knowledge of subject matter.
7. Tests are sufficiently flexible in nature to provide for the instructor an opportunity to present remedial work when and where needed.
8. The amount of class time needed is not excessive and is devoted largely to class discussion.
9. The results of the testing provide the instructor with an additional check on the accomplishment of the individual.
10. The efficiency of the testing program is greatly enhanced by the approval of the pupil.

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Should Children Help Each Other in Arithmetic?

By F. B. RIGGS
Indian Mountain School, Lakeville, Connecticut

BOYS AND GIRLS often help each other in their school work. Is this help beneficial to the pupil? Let us confine our question to the skills of arithmetic, such as addition, subtraction, multiplication, division of the whole numbers and fractions, and the operations of percentage, exponents, and the like.

In favor of the proposition that pupils should help each other the following considerations are offered:

1. Why examine the source if the resulting answers are correct? Practice in obtaining the right answer is to be desired no matter by what method that practice is obtained.

2. Some pupils learn certain arithmetical procedures better from other pupils than from teachers.

3. The pupils will get help from other pupils anyway, so there is no use in preventing them.

Opposed to the proposition these considerations are offered:

1. Do pupils in teaching their peers, ever give the "why" of a procedure? Are they not content to give the "how"? The "why" is fundamental, the "how" is not. "Hows" depend largely on visual or auditory memory; "whys" depend on thorough understanding. One "why" paves the way for other "whys." One "how" has no such transcendant value. For example: The "how" boy learns to "do" $a^3 \times a^4 = a^7$. He has *remembered* to add exponents. He learns to "do" $\frac{a^4}{a^3} = a$. He has *remembered* to subtract exponents. He may easily forget when to add and when to subtract. The "why" boy, when in doubt could write then $\frac{a^4}{a^3} = \frac{a \times a \times a \times a}{a \times a \times a}$. Oh, yes, subtract exponents! The "why" has produced the "how" independent of memorizing a fact, which, be it remembered, is not in daily use.

A parallel can be drawn, up to and including engineering practice. The "why" principle in the engineer, distinguishes him from the draftsman who has never emerged from the "how" plateau.

2. The habit of the pupils helping each other must later be abandoned. Most schools are apt to consider any help received from other pupils in the skills of mathematics as a form of cheating.

3. How can a teacher really check a pupil's written work? Is he not really checking the work of some boy who is not the pupil in question?

4. Pupils who do not receive help from their friends and pupils who do receive such help (some of which is usually without the knowledge of their teacher) are put in a relatively inequitable position toward each other and toward their teachers, when competition comes into play.

In conclusion it would appear that the question is not merely "how can the pupil best learn these skills?" but also, "What permanent habits is he forming?"

PLAYS

Back numbers of *The Mathematics Teacher* containing the following plays are available and may be had from the office of *The Mathematics Teacher,* 525 West 120th Street, New York.

A Near Tragedy. Miller, Florence Brooks, XXII, Dec. 1929.
An Idea That Paid. Miller, Florence Brooks, XXV, Dec. 1932.
If. Snyder, Ruth L. XXII, Dec. 1929.
Mathematical Nightmare. Skerrett, Josephine, XXII, Nov. 1929.
Mathesis. Brownell, Ella, XX, Dec. 1927.
The Eternal Triangle. Raftery, Gerald, XXVI, Feb. 1933.
The Mathematics Club Meets. Pitcher, Wilimina Everett, XXIV, April 1931.

Price: 40¢ each or 25¢ in orders of four or more.

The Graphical Interpretation of Equations Involving Reciprocals

By MAURICE L. HARTUNG
University High School, Madison, Wisconsin

THERE ARE certain types of problems in algebra which lead to a system of simultaneous equations of the form

$$\frac{a}{x} + \frac{b}{y} = c$$

(1)

$$\frac{d}{x} + \frac{e}{y} = f,$$

in which x and y are unknown. Granted that such problems occur but rarely and in many cases are artificial, I trust it is safe to assume that many teachers and pupils still find interest in their solution. If this is true, then teachers at least should be interested in studying the *nature* of this solution a little more thoroughly than is usually done.

Since the system (1) is linear in $\frac{1}{x}$ and $\frac{1}{y}$, is it generally solved for these expressions by one of the methods used in solving linear systems. Thus if both members of the first equation are multiplied by d, both members of the second by $-a$, and the results are added, an equation which may be easily solved for y is obtained. With a little guidance from textbook or teacher, a pupil can find the solution

(2) $$x = \frac{ae - bd}{ce - bf}, \qquad y = \frac{ae - bd}{af - cd}.$$

Occasionally, however, one finds a student who clears of fractions! The equations of the system

(3)

$$ay + bx = cxy$$
$$dy + ex = fxy$$

so obtained are quadratic, and the teacher generally advises the pupil that his attack is "wrong," starts him off in the "right" direction, and

dismisses the matter. But if the systems (1) and (3) are equivalent the solution of (1) may be also found by solving the system (3) of simultaneous quadratic equations. Such systems in general have four solutions. The first method indicated gave us but one, namely equations (2). What has become of the other three? Is it possible that we have lost them in solving (1) by an elementary method?

In considering the answer to these questions the graphical interpretation of system (3) is of interest. It is shown in Analytic Geometry that the general quadratic equation in two variables

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{4}$$

is an hyperbola, parabola, or ellipse according as $B^2 - 4AC$ is greater than, equal to, or less than zero. Since in the first equation of (3) we have

$$A = C = 0, \quad B = c, \quad \text{and } B^2 - 4AC = c^2,$$

it follows that the curve is in general an hyperbola. Degenerate conics occur when a, b, or c is zero, in which case the equation represents straight lines. The second equation of (3) is of the same type, and since $x = 0$, $y = 0$ satisfies both equations, it is clear that, in general, the system (3) represents two hyperbolas intersecting *at the origin* as well as at the point given by equations (2). It may also be shown that the hyperbolas are equilateral with axes rotated 45° with respect to the coordinate axes. When the system (3) is solved by eliminating one of the variables a quadratic equation is obtained. Since solving two quadratic equations simultaneously leads in general to a quartic equation, this means that two of the four intersections are at infinity.

Then the solutions of system (3) are $x = 0$, $y = 0$; the values of x and y given by formulas (2), and two solutions at infinity. We may not say, however, that system (1) has the same solutions as system (3), for (1) is meaningless at $x = 0$, $y = 0$. Graphically, the curves of system (1) are the same as those of system (3), *except that they have a "hole" or discontinuity at the origin.* Thus the equations (2) found by the methods used for linear systems actually give the only finite solution of system (1). But is it not true that in the effort to obtain these numbers, the interesting graphical interpretation of the situation is generally overlooked? After all, "hyperbolas" with "holes" in them are not very common!

Some Questions on Methods of Teaching for a Teacher's Own Self-Analysis

By ALAN D. CAMPBELL
Syracuse University

THE FOLLOWING LIST of questions was given to the graduate assistants in the mathematics department of the College of Liberal Arts of Syracuse University. These questions were designed to make a young teacher indulge in some self-analysis. Perhaps these questions will be of interest to the readers of THE MATHEMATICS TEACHER. The questions are as follows:

"Graduate Assistants:—

"Consult still more with Professors ———, ———, *and* ———.

"Show them your tests, your daily assignments, how you occupy the class period, how much class work you have the students do, how much home-work. Ask them if you may use one of their tests.

"Do you explain all the work yourself? Do you have students explain problems at the board? Do you talk (lecture) too much? Do you go slowly enough?

"Are you keeping in step with the above teachers? Do you plan with them how far you will go the next week? Have you asked one of them to visit your classes and to criticize? Have you asked their permission to visit their classes?

"When you explain anything do you make sure that every student in the class understands it? Is your board work neat and arranged nicely? Can the students see your board work? Do you give two sets of tests so as to guard against cribbing? Do the students get you started on subjects outside the lesson? Do you explain the lesson ahead of time? Do you have several ways of explaining something, so that you can reach every student?

"Is your voice loud enough, does your voice put the back row to sleep? Are you enthusiastic about the subjects you teach? Do you have any disorder, such as throwing of chalk, whispering, and the like? Do you go over with the class any problems that are worked out in the text? Do you compel *students who are failing to come and*

see you? Are you getting the students to ask questions? Do you look on the class period as a lecture period, a laboratory period, or a recitation period?

"Do you permit part of your class to be idle (because there is not enough blackboard space) or do you hand out single sheets of paper upon which they may work? Do your students study other lessons in your class, or write themes, and the like? Do you use problems outside the textbook you are using?

"As to your own graduate work, how are you impressing your teachers? Are you going to sleep in their classes? Are you asking questions? You might ask the above teachers how to study for graduate courses."

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Should Mathematics Teachers Seek to Know the Needs of Society?

By MERTON TAYLOR GOODRICH, M.A.
State Normal School, Keene, New Hampshire

THE ARTICLE "Improving America's Mathematics" in the issue of last May of *The Mathematics Teacher* opened with an illustration so striking that its fallacy was well hidden. It stated in substance that a doctor does not go out on the street and ask different people what kind of pills they have been taking in order to prescribe for his patients. There are many who do not agree with the inferences and conclusions made in that article from that clever but obvious statement.

In the first place the schoolroom is not a sick-room. It is not the pupil who is ill. It is society that is sick of the textbook and schoolroom mathematics that is not correlated with social usage. If an analogy between educators and physicians is to be employed, the patient is society. About the only way of applying educational remedies to society is by teaching pupils in the right way.

Educators are trying to find the right way to teach mathematics. They are making a diagnosis of the condition of society. They are asking what is the situation that actually obtains in the world in regard to the use of mathematics. They are asking of the man in the street not what mathematics to teach but what mathematics does he actually use.

Just as the decision of what to prescribe after the diagnosis of a case rests with the doctor, so the decision of what to teach rests with the educator. The basis for this decision should be the conditions he finds in society. It is as justifiable for a teacher to ask questions of society as for a physician to ask questions of his patient.

One of the chief difficulties with the school situation is that too often teachers have prescribed courses without knowing the needs of society. Society knows what kind of mathematics it uses. Teachers can not justify the claim that they know more about what kind of mathematics society is using than society does itself. It is up to the teacher to find out what are the useful and valuable units of mathematics and the limitations of these units as used by society. This

can be done only by direct inquiry of people who are engaged in the activities of the world and by direct observation. This means a survey not of what teachers think is being done or what they believe ought to be done. It means a survey of those units in mathematics that are actually used by the housewife, the mechanic, the chemist, the merchant, the banker, the engineer, in fact by everyone whether he be a specialist or not. It means a diagnosis not only of the units that are used but of how much they are used and the sizes of the numbers in use. When the teacher has found out from society what the conditions are in society, then and then only can the teacher use sound judgment in deciding what to teach.

A teacher has no more right to require a pupil to take a unit of arithmetic for which he has no need than a physician has to prescribe a pill for which his patient has no need. Just as good physicians seek to prescribe remedies and as much of each as will be of greatest benefit to their patients, leading teachers of mathematics after careful diagnoses of society are seeking to prescribe those units and as much of each as will be of greatest benefit to society. Just as physicians for years have been justified in making diagnoses of the condition of their patients, educators are justified in making a diagnosis of the condition of the world in regard to mathematics.

Why Mathematics in the High School

Every mechanical, scientific, financial, productive, distributive function of our social structure is based on mathematical thinking and mathematical thought.

Whenever, anywhere, men press on from mere floundering in an enterprise to certain prediction they are thinking mathematically. This is the most significant result of mathematics teaching. Mathematics must be a component of *everyone's* education.

The mass predicates its thinking on second grade arithmetic; its judgments must be based on a consciousness of relations, and the powers and abilities to seek relations and predict outcomes. This will result when every child is properly trained in mathematics.

Edmund Halley

Born in London, November 8, 1656
Died at Greenwich, January 14, 1742

"While we thought that the eulogium of an astronomer, a physicist, a scholar and a philosopher comprehended our whole subject, we have been insensibly surprised into the history of an excellent mariner, an illustrious traveller, an able engineer, and almost a statesman."[1]

HALLEY was the only son of a wealthy Londoner. He distinguished himself at St. Paul's School and when he entered Queen's College at Oxford in 1673 he was acquainted with Greek, Latin, and Hebrew. He later had occasion to study Arabic. He was greatly interested in astronomy and before he was twenty he sent to the Royal Society a treatise which had the title *Direct and Geometrical Method of finding Aphelia and Eccentricity of Planets.*

In 1676, he persuaded his father to let him leave the university and to finance a trip to catalogue stars in the southern hemisphere thus supplementing work then in progress in England and on the continent. Charles II made favorable arrangements with the East India Company and Halley sailed to St. Helena where he stayed for over a year and a half. The climate proved unsuitable for astronomical observations, but Halley succeeded in cataloging more than 300 stars. This work earned him the title of the "southern Tycho."[2]

On his return to England, Halley presented a planisphere of the southern hemisphere to the king and gave his catalogue of stars to the Royal Society. The University of Oxford granted him his master's degree in recognition of this work and the Royal Society made him a member.

During the same time, Halley had improved the sextant, he had observed a transit of Mercury, and he had determined the length of the second's pendulum at the equator. This last piece of work is

[1] Ortous de Mairan. Quoted from the *Dictionary of National Biography* under "Edmund Halley," on which this article is chiefly based. The reader is also referred to the *Correspondence and Papers of Edmond Halley preceded by an Unpublished Memoir of His Life by One of His Contemporaries and the "Eloge" by d'Ortous de Mairan.* Arranged and edited by Eugene Fairfield MacPike. History of Science Publications, New Series, II, Oxford, 1932.

[2] Tycho Brahe was the Danish astronomer from whose observations Kepler derived his laws of planetary motion.

important in the history of weights and measures. The idea of using the length of the second's pendulum either as a unit of length or as the source of a unit of length seems to have originated with the Royal Society about 1670. In 1673, however, it was discovered that the length of the second's pendulum varied at different parts of the earth's surface, indicating that the earth was not a true sphere. Measurements were made in Peru and in Lapland to determine its real shape. Newton believed that the earth was flattened at the poles: the Cassinis in France held that it was elongated at the poles. The work in Lapland that substantiated Newton's conjecture was said to have "flattened out both the earth and the Cassinis." At the time of Halley's observations, the validity of the variation of the second's pendulum was somewhat in question. His work did much to substantiate the truth of the matter. The second's pendulum as a unit of length was not given up until the members of the French Academy decided to undertake the great survey across France on which the length of the meter was to be based.

From 1678 to 1684, Halley traveled in Europe and continued his work in astronomy.

A question put by Halley to Newton in 1684 led to Newton's reconsideration of some earlier calculations and to the writing of the *Principia*. The publication funds of the Royal Society had been used in printing a work on fishes and the Society accordingly authorized the publication of Newton's work with the statement that it was to be done at Halley's expense. As Halley was no longer a wealthy man and as he was occupied with his own researches, his sacrifice of time and money is very significant.

In 1693, Halley published his study of the mortality tables of Breslau, a city whose records were unusually complete. From these tables, he concluded that the price of an annuity should vary according to the age of the person for whom it was purchased. These tables earned him the title of the virtual originator of the science of life statistics. His attempt to regulate the price of annuities according to the age of the annuitant was received with much skepticism.[3]

In 1696 through Newton's influence, Halley was made deputy controller of the mint in Chester. While there, he worked on the determination of heights of mountains by a barometer. He had a

[3] Cf. a quotation from John Ward, *Young Mathematician's Guide*, 1758 ed., given in a *Short History of Mathematics*, pp. 138–139.

theory that the longitude of a place at sea might be determined by the variation in the compass from the true north. He interested William III in this project and was given a ship and spent nearly two years in this study. In 1701, he published a chart which showed the lines of equal declination of the compass from the true north which were later called "Halleyan lines." The *Dictionary of National Biography* intimates that this was the first use of this graphic device for presenting a great mass of data. Thus Halley is connected with the beginnings of physical geography and with the isotherms, isobars, etc. that are familiar to us all.

In the same year, Halley made a study of the tides in the English Channel and at the request of the Emperor he surveyed the ports in the Adriatic Sea.

On the death of Wallis (1703), he was appointed Savilian Professor of Geometry at Oxford, a position which he held until 1721. It was during this period that he published the first complete edition of the *Conics* of Apollonius, and made a number of discoveries in mathematics. He seems to have been the first to determine a conic when the focus and three points on the conic are given.

From 1713 to 1721 he was secretary of the Royal Society, resigning when he was appointed Astronomer Royal succeeding Flamstead. He held this position for the remainder of his life.

Halley's name is best known to the non-mathematical world through his work with comets. He computed the orbits of a number of them and became convinced that the comet of 1682 had a period of about 76 years. He predicted the return of this comet in 1758 and in recognition of this pioneer work, the comet is called by his name.

Halley's career differs from those of many of his contemporaries in its freedom from controversy over priority of inventions. This is surprising for he was known to be independent in thought and free in expresing his opinions. It is suspected that Halley is the "infidel mathematician" to whom Berkeley addressed the *Analyst* (1734). This was written with the purpose of inquiring into the methods of mathematicians of the first part of the eighteenth century with the same freedom which the person to whom it is addressed was treating theology. After putting some pointed queries in regard to fluxions which he describes in interesting terms and which are later to be dubbed the "ghosts of departed quanti-

ties," Berkeley states that "he who can digest a second or third fluxion, a second or third difference, need not, methinks, be squeamish about any point in divinity."[4]

If Halley was indeed the cause of Berkeley's diatribe, we are indebted to him not only for the publication of the *Principia* but, indirectly, for the publication of an essay which set English mathematicians to work building a firmer foundation for Newton's calculus.

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[4] See a *Source Book in Mathematics*, edited by David Eugene Smith, New York, 1929, 627–634.

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NEWS NOTES

THE FOLLOWING programs were presented at the Oklahoma Educational Association in Tulsa, Oklahoma, on February 3, 1933.

High School Section—Chairman, Kate Barbour, Norman; Secretary, Mattie E. Bogue, Okmulgee:

"The Relation of High School Mathematics to College Mathematics," Dr. W. E. Howard, Tulsa.

"Advantages of the Group Method of Instruction and the Use of Models," Mrs. Verdie Goss, Heavener.

"Some Important Concepts in Algebra," Dora McFarland, Norman.

"Problems Confronting High School Teachers of Mathematics in Oklahoma," George D. Hann, Clinton.

"Mathematics Clubs and Their Possibilities in High School," Dr. Kathryn Wyant, Tahlequah.

College Section—Chairman, Clarence McCormick, Weatherford; Secretary, A. M. Wallace, Edmond:

"Comparison of Mathematics as Taught in Institutions of Higher Education in the United States and Russia," Nicholas M. Oboukhoff, Stillwater.

"What Constitutes a Good College Mathematics Teacher," G. Emory Meador, Oklahoma City.

"Some Desirable Objectives in Teaching College Mathematics and Methods of Attaining Them," J. O. Hassler, Norman.

A committee to articulate college and high school, investigating the cause of failures in college mathematics consists of E. F. Allen, A. and M. College, Stillwater, chairman; Clarence McCormick, Teachers College, Weatherford; Mattie E. Bogue, Okmulgee; Martha Denny, Oklahoma City.

The officers of the Mathematics Section for the coming year are: W. B. Wise, Woodrow Wilson Junior High School, Tulsa, chairman; Martha Denny, Classer High School, Oklahoma City, Secretary.

The third meeting of the Cleveland Mathematics Club for the year (1932–33) was held on March 4th, the speaker being Professor C. N. Stokes of Temple University, Philadelphia.

The Nineteenth Annual Meeting of the Kansas Section of the Mathematical Association of America and Twenty-ninth Annual Meeting of the Kansas Association of Mathematics Teachers was held at the Topeka High School on February 11th. The program follows:

Forenoon session, 10:00 A.M., Room 318. Joint Session—K. A. M. T. and M. A. A.:

"Homer Myers—An Appreciation," W. H. Garrett, Baker University.

"Sidelights on the International Congress of Mathematicians, Zurich, 1933." U. G. Mitchell, University of Kansas.

Afternoon Session, 1:15 P.M., Room 318. K. A. M. T.:

Business Meeting: Report of the Secretary-Treasurer; Report of Nominating Committee and Election of Officers; Report of the Committee on "Reinstatement of Mathematics as a Required High School Subject," W. T. Stratton, Kansas State College.

"How Much Arithmetic Should be Taught in the Elementary School," A. W. Philips, Kansas State Teachers College of Emporia.

"Mathematics Clubs," Nina McLatchey, Topeka High School.

"The Situation as Regards Secondary School Mathematics," Emma Hyde, Kansas State College.

"Application of Dr. Swenson's Methods," Bird M. Weimer, Wichita High School, North.

Afternoon Session, 1:15 P.M., Room 319. *M. A. A.*:

Business Session.

"Some Properties of Orthogonal Functions," R. W. Babcock, Dean of the Division of General Science, Kansas State College.

"Certain Loci Related to a Variable Triangle," Neil F. Shell, Formerly Professor of Mathematics in the Fort Smith, Ark., Junior College.

"Generalized Coordinates and the Lagrangian Equations of Motion," R. G. Smith, Kansas State Teachers College of Pittsburgh.

"The Early Life of Carl Friedrich Gauss," G. Waldo Dunnington, Horner College, Kansas City, Mo.

Officers of the Kansas Section, of the Mathematical Association of America:

Chairman, Oscar J. Peterson, Kansas State Teachers College of Emporia.

Vice-Chairman, W. A. Harshbarger, Washburn, College.

Sec.-Treas., Lucy T. Dougherty, Kansas City Junior College.

Officers of the Kansas Association of Mathematics Teachers:

President, Edith D. Whitted, Newton. Vice-Presidents, Edna Austin, Topeka, Marie Dale, Dode City, M. L. Dubach, Columbus, Mary Lloyd, Atchison, Kathleen O'Donnell, Salina, Ruth Payne, Wellington.

Sec.-Treas., Vera Steininger, Abilene.

Editor of the Bulletin, Ina E. Holroyd, Kansas State College.

The following program for this year of the Mathematics Club of the University of Kansas which was organized in 1911 will be of interest to our readers:

October 3: Business Meeting.

October 17: "On the Square," Prof. G. W. Smith.

October 31: "Primitive Methods of Measuring Time," Ralph L. Scott.

November 14: "Mathematics and Astronomy," Prof. W. H. Garrett, Baker University.

December 5: "Ballistics and Gunnery," Major W. C. Koenig.

January 9: "Dimensional Analysis," Daniel P. Johnson.

January 23: "Some Chemical Applications of Mathematics," Prof. A. W. Davidson.

February 13: "Nine Point Circle," Lilly E. Somers.

February 27: "Higher Plane Curves," Dorothy E. Doering.

March 13: "Correlation in Connection with Forecasting," Prof. E. B. Dade.

March 27: "Mathematics in Music," Prof. C. S. Skilton.

April 10: "Finite Geometry," Eula Johnson.

April 24: "Mathematics in Scientific Museums," Prof. U. G. Mitchell.

May 15: Picnic with Mathematics Clubs of Neighboring Colleges as Invited guests.

Officers: President, Oleta M. Markham; Vice-President, Otis F. Brubaker; Secretary-Treasurer, Walt R. Simmons; Social Chairman, Carol Hunter; Faculty Adviser, Prof. J. J. Wheeler.

Program Committee, Albert S. Palmerlee, Chairman, Gilbert Ulmer, Elizabeth Perry.

NEW BOOKS

My First Drill Book in Numbers. For the second grade, by C. L. Thiele, supervisor of exact sciences, Detroit Public Schools and Irene Sauble, assistant supervisor of exact sciences, Detroit Public Schools, illustrated by Minnie Hansen Rousseff. Rand McNally & Company, Chicago, Ill., 1932. 102 pages. List price 36 cents.

An extraordinary little book designed to make youngsters actually like arithmetic. Adaptable to any basic text or method of teaching, it is prepared in a three-step-to-mastery plan—STUDY, PRACTICE, TEST. First, little tots are shown how to study. Then they are given actual practice devoid of all dull number routine. The content is based on the common experiences of the child in the home, in the school, and on the playground. The child picks flowers, goes to the toy shop, eats cookies, counts goldfish, attends a Halloween party, compares ages, blows soap bubbles, feeds rabbits, and does all kinds of other things so dear to the heart of a second grader. Then interesting timed tests show youngsters how well they have learned. My First Drill Book in Numbers makes number mastery as easy and fascinating as play.

An Analytical Study of Mathematical Abilities. By George Joseph Cairns. Saint Mary College and Academy, Monroe, Mich.

This publication is a Ph.D. dissertation which will be of interest to those students and teachers of mathematics. It is a study in method in the hope that the technique employed therein may help to bring nearer together the conflicting views as to the nature of mathematical ability.

Unit Mastery Arithmetic. By John C. Stone and Clifford N. Mills. Three book series; primary, intermediate and advanced. Benj. H. Sanborn Company, 1932.

The authors have attempted to present the subject of arithmetic from the very first as a thought subject rather than as a mere skill subject. They claim ten distinguishing features as follows:

1. The work of each grade is divided into ten units. Each unit consists of a single skill, or closely related skills, in the lower grades; and with a single topic, or closely related topics, in the upper grades.

2. Each page is a complete sub-unit. No problems or drills are carried over to another page.

3. The practice work in the lower grades is scientifically made, strategically placed, and abundant in quantity.

4. The problem-solving side of arithmetic receives far greater attention in these books than in any other textbook on the market.

5. The presentation of each new process begins with a concrete problem and ends with problems.

6. The vocabulary employed has been carefully graded and kept within the range of the child's understanding.

7. The development in the "two-step" problem in the Fourth-Grade work receives unusual emphasis.

8. Practice work to maintain skills

in computation is given in each unit throughout the entire series.

9. A series of semi-semester tests, published in pamphlet form and based directly on the text, offers the teacher or superintendent frequent opportunity to check up on the child's progress.

10. The general attractiveness of the series, as to binding, color illustrations, appearance of the page, humor and liveliness of the content.

Teachers of arithmetic will be interested in this series.

Objective Exercises in Units of Plane Geometry. By William W. Strader and Lawrence D. Rhoads, Dickinson High School, Jersey City. 100 pages. Published by The John C. Winston Company, Philadelphia. List price, 48 cents.

Most geometry teachers agree that standardized diagnostic and remedial exercises, which have proved so valuable in other subjects, are not particularly effective in geometry. Pupil differences are too varied and far-reaching to respond to generalized treatment. The authors of this workbook, who are also the authors of the well-known Strader and Rhoads Geometries, have developed in their own classrooms this program of objective exercises, consisting mainly of developmental and practice work. The exercises are simple, requiring the pupil to work with fundamentals only, but even the slowest pupils will learn geometry because they actually *do* geometry in answering the exercises. There are also general diagnostic exercises, which have both diagnostic and learning value. Comprehensive Achievement Exercises are printed in a separate section, separately bound, so they may be used for testing if desired. All exercises are completely objective, and the answers in the Teacher's Manual and Key are so arranged that scoring is a matter of seconds. Because of the unit arrangement this workbook may be used with any basal text.

Plane Geometry. By Mabel Sykes, instructor in mathematics, Bowen High School, Chicago, Ill., Clarence E. Comstock, professor of mathematics, Bradley Polytechnic Institute, and Charles M. Austin, head of the department of mathematics, Oak Park and River Forest Township High School, Oak Park, Ill. Rand McNally & Company, Chicago, Ill., 1932. 460 pages. List price $1.28.

This textbook humanizes geometry, emphasizing its beauty and usefulness. Geometric reasoning is made more enjoyable by showing how it fits in with life experiences and situations. Clear and orderly thinking is stressed throughout. In connection with each exercise and theorem, there is given a brief statement of the plan or analysis to be used in solving the exercise or theorem. It gives the student at the outset the gist of the argument, calls his attention to the method of proof employed, and gives him some idea of how the proof may have been thought out originally. The material is arranged for easy adaptability to classes and pupils of varying ability as well as to any teaching procedure. In addition to a "minimum essentials" course, there is additional work for brighter pupils. Summaries, special review exercises and survey tests add to the usability of the textbook.